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An Intrinsic Characterization of Lower Semicontinuity of the Metric Projection in $C_0(T, X)$

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We give an intrinsic characterization of finite-dimensional subspaces G in $C_0(T, X)$ whose metric projection P_G is lower semicontinuous. Namely: P_G is lower semicontinuous if and only if, for every nonzero element g in G,

$$\operatorname{card}(\operatorname{bd} Z(g)) \leqslant \dim\{p \in G: \operatorname{int} Z(g) \subset Z(p)\} - 1.$$

1. Introduction

Let T be a locally compact Hausdorff space and X a strictly convex Banach space. Let $C_0(T, X)$ be the Banach space of continuous mappings f from T to X which vanish at infinity; i.e., the set $\{t \in T: \|f(t)\|_X \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$. $C_0(T, X)$ is endowed with the supremum norm:

$$\|f\| = \sup\{\|f(t)\|_X: t \in T\},$$

where $\|\cdot\|_X$ denotes the norm on X.

For any set G in $C_0(T, X)$ and f in $C_0(T, X)$, define

$$d(f, G) = \inf\{\|f - g\| : g \in G\},$$

$$P_G(f) = \{g \in G: \|f - g\| = d(f, G)\}.$$

Here P_G is called the metric projection from $C_0(T, X)$ onto G.

Recall that P_G is said to be lower semicontinuous (lsc) at f if and only if for any $g \in P_G(f)$, $\lim_{n\to\infty} f_n = f$ implies $\lim_{n\to\infty} d(g, P_G(f_n)) = 0$. P_G is said to be lsc if and only if P_G is lsc at every f in $C_0(T, X)$.

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The lower semicontinuity of P_G is closely related to the existence of continuous selections of P_G. By the well-known Michael selection theorem [14], we know that for a finite-dimensional subspace G, the lower semicontinuity of P_G implies that there is a continuous selection of P_G; i.e., there is a continuous mapping S from $C_0(T, X)$ to G such that $S(f) \in P_G(f)$ for all f in $C_0(T, X)$.

There has been a nice characterization of the lower semicontinuity of P_G. The result is as follows:

THEOREM A. *Suppose that G is a finite-dimensional subspace of $C_0(T, X)$. Then P_G is lsc if and only if for each f in $C_0(T, X)$, the set $\{t \in T\colon p(t) = g(t)$ for all $p, g \in P_G(f)\}$ is open.*

This theorem was first obtained by Blatter, *et al.* when X is the Banach space R of real numbers [1], and was generalized by Brosowski and Wegmann to arbitrary strictly convex X later [3].

On the other hand, there are many works studying the continuous selections of P_G [4–7, 9–13, 15–18]. Recently, some new progress has been made. Here are two results which are relevant to this paper.

THEOREM B. *Suppose that G is a finite-dimensional subspace of $C_0(T, R)$. Then P_G has a continuous selection if and only if for every f in $C_0(T, R)$, there is an element g in $P_G(f)$ such that for every p in $P_G(f)$,*

$$\operatorname{int}\{t \in T\colon (f(t) - g(t)) \cdot (g(t) - p(t)) \geqslant 0\} \supset (f - P_G(f)),$$

where $E(f - P_G(f)) := \{t \in T\colon |f(t) - p(t)| = d(f, G)$ for all $p \in P_G(f)\}$.

THEOREM C. *Suppose that T is a compact and locally connected Hausdorff space and that G is a finite-dimensional subspace of $C_0(T, R)$. Then P_G has a continuous selection if and only if every nonzero g in G satisfies the following two conditions:*

(i) $\operatorname{card}(\operatorname{bd} Z(g)) \leqslant \dim\{p \in G\colon \operatorname{int} Z(g) \subset Z(p)\} =: r(g)$,

(ii) *g has at most $r(g) - 1$ zeros with sign changes.*

Here $Z(g)$ is the set of all zeros of g and $\operatorname{card}(\operatorname{bd} Z(g))$ denotes the cardinal number of the boundary $\operatorname{bd} Z(g)$ of $Z(g)$.

The necessity of Theorem B was given by Lazar *et al.* [9] and Brown [4]. The sufficiency of Theorem B was proved recently by the author [10] (or see [7, 11]). It was first discovered by Nürnberger and Sommer that one can characterize a finite-dimensional subspace G of $C[a, b]$ whose metric projection P_G has continuous selections by the zero sets of elements in G [15–18]. Motivated by their works, the author established Theorem C [12] (or see [6, 7, 11]).

Theorem B is of the same nature as Theorem A, i.e., the characterization conditions are both involved with the metric projection P_G, which is difficult to determine in general. Encouraged by Theorem C, we try to give an intrinsic characterization condition, similar to that in Theorem C, which ensures the lower semicontinuity of P_G. This is the main purpose of the present paper.

Fortunately, we obtain the following simple form of the characterization condition for the lower semicontinuity of P_G.

THEOREM 1. *Suppose that G is a finite-dimensional subspace of $C_0(T, X)$. Then P_G is lsc if and only if for every nonzero g in G,*

$$\operatorname{card}(\operatorname{bd} Z(g)) \leqslant (\dim X)^{-1} \cdot \dim\{p\,|_{\operatorname{bd} Z(g)} : p \in G \text{ and } \operatorname{int} Z(g) \subset Z(p)\}, \tag{1.1}$$

where $p\,|_{\operatorname{bd} Z(g)}$ denotes the restriction of p to $\operatorname{bd} Z(g)$.

THEOREM 2. *Suppose that G is a finite-dimensional subspace of $C_0(T, R)$. Then P_G is lsc if and only if for each nonzero g in G,*

$$\operatorname{card}(\operatorname{bd} Z(g)) \leqslant \dim\{p \in G : \operatorname{int} Z(g) \subset Z(p)\} - 1. \tag{1.2}$$

COROLLARY 1. *Suppose that G is a finite-dimensional subspace of $C_0(T, R)$. If each nonzero g in G satisfies* (1.2), *then P_G has a continuous selection.*

In Section 2, we give the proofs of the above results. In Section 3, we give some remarks about the results discussed in this section, including a counterexample which shows that Theorem C fails to be true if T is not locally connected.

2. PROOF OF MAIN RESULTS

From now on, we always assume that G is a finite-dimensional subspace of $C_0(T, X)$ and that X is strictly convex. Our proof is based on the following pointwise version of Theorem A.

LEMMA 1. *P_G is lsc at f if and only if*

$$E(f - P_G(f)) \subset \operatorname{int}\{t \in T : p(t) = g(t) \text{ for all } p, g \in P_G(f)\}.$$

Lemma 1 was announced in [2] as an unpublished theorem of Blatter in the case of $X = R$. The author generalized Blatter's result in [13].

Before we prove Theorem 1, we need several technical lemmas.

LEMMA 2. *Suppose that* $t_i \in T$ *and* $\varphi_i \in X^* \setminus \{0\}$, $1 \leqslant i \leqslant r$, *satisfy*

$$\sum_{i=1}^{r} \varphi_i(g(t_i)) = 0, \qquad g \in G.$$

Then for any $f \in C_0(T, X)$ *with* $\varphi_i(f(t_i)) = \|\varphi_i\| \cdot \|f\|$, $1 \leqslant i \leqslant r$, *we have*

$$d(f, G) = \|f\|,$$
$$t_i \in E(f - P_G(f)), \qquad 1 \leqslant i \leqslant r. \tag{2.1}$$

Proof. For any $g \in P_G(f)$, we have

$$\begin{aligned}\sum_{i=1}^{r} \|\varphi_i\| \cdot d(f, G) &\leqslant \sum_{i=1}^{r} \|\varphi_i\| \cdot \|f\| \leqslant \sum_{i=1}^{r} \varphi_i(f(t_i)) \\ &= \sum_{i=1}^{r} \varphi_i(f(t_i) - g(t_i)) \leqslant \sum_{i=1}^{r} \|\varphi_i\| \cdot \|f(t_i) - g(t_i)\|_X \\ &\leqslant \sum_{i=1}^{r} \|\varphi_i\| \cdot \|f - g\| = \sum_{i=1}^{r} \|\varphi_i\| \cdot d(f, G).\end{aligned}$$

Thus equality must hold throughout this string of inequalities, i.e.,

$$d(f, G) = \|f\|,$$
$$\|f(t_i) - g(t_i)\|_X = \|f - g\| = d(f, G), \qquad 1 \leqslant i \leqslant r,\ g \in P_G(f). \tag{2.2}$$

Note that (2.2) implies (2.1). ∎

Remark. We are indebted to Professor F. Deutsch for the proof of Lemma 2, which simplifies our original proof.

LEMMA 3. *Suppose that* V *is a subset of* T *and* $G_0 := \{p \in G : V \subset Z(p)\}$. *If* $A \subset T \setminus V$ *satisfies*

$$\operatorname{card}(A) > \dim G_0|_A / \dim X, \tag{2.3}$$

then there exist $t_i \in A \cup V$ *and* $\varphi_i \in X^* \setminus \{0\}$, $1 \leqslant i \leqslant r$, *such that*

$$\sum_{i=1}^{r} \varphi_i(g(t_i)) = 0, \qquad g \in G,$$
$$\{t_i : 1 \leqslant i \leqslant r\} \cap A \neq \varnothing.$$

Proof. If $\dim X$ is infinite, then for any $t_0 \in A$, since $\{g(t_0): g \in G\}$ is a finite-dimensional subspace of X, there is $\varphi \in X^* \backslash \{0\}$ such that

$$\varphi(g(t_0)) = 0, \qquad g \in G.$$

So Lemma 3 is true if $\dim X$ is infinite.

Now assume that $\dim X = n < \infty$. Then $X^* = n$. Since $\dim G < \infty$, we can select $t_i \in V$, $1 \leqslant i \leqslant s$, and $\varphi_{ij} \in X^* \backslash \{0\}$, $1 \leqslant j \leqslant m_i$, $1 \leqslant i \leqslant s$, such that

$$\dim G|_V = \sum_{i=1}^{s} m_i, \tag{2.4}$$

and $\{\varphi_{ij} \circ \delta_{t_i}: 1 \leqslant j \leqslant m_i, 1 \leqslant i \leqslant s\}$ is a linearly independent system on G; i.e., if

$$\sum_{i=1}^{s} \sum_{j=1}^{m_i} c_{ij} \cdot \varphi_{ij}(g(t_i)) = 0, \qquad g \in G,$$

then

$$c_{ij} = 0, \qquad 1 \leqslant j \leqslant m_i, \; 1 \leqslant i \leqslant s.$$

Suppose that $X^* = \operatorname{span}\{\psi_i: 1 \leqslant i \leqslant n\}$ and $A = \{t_i: s+1 \leqslant i \leqslant r\}$. Set

$$\Phi = \{\varphi_{ij} \circ \delta_{t_i}, \psi_k \circ \delta_{t_l}: 1 \leqslant j \leqslant m_i, 1 \leqslant i \leqslant s, 1 \leqslant k \leqslant n, s+1 \leqslant l \leqslant r\}.$$

Then $\Phi \subset (G|_{A \cup V})^*$ and from (2.3), (2.4) we obtain

$$\begin{aligned} \operatorname{card}(\Phi) &= \sum_{i=1}^{s} m_i + (r-s)n = \dim G|_V + \operatorname{card}(A) \cdot \dim X \\ &> \dim G|_V + G_0|_A = \dim G|_{A \cup V}. \end{aligned}$$

Thus there is a nonzero linear combination φ of elements in Φ such that

$$\varphi(g) = 0, \qquad g \in G|_{A \cup V}. \tag{2.5}$$

Obviously, φ may be represented as

$$\varphi(g) = \sum_{i \in I} \varphi_i(g(t_i)), \qquad g \in G, \tag{2.6}$$

where $I \subset \{i: 1 \leqslant i \leqslant r\}$ is nonempty, $\varphi_i \in X^* \backslash \{0\}$ for $i \in I$, and

$$\varphi_i \in \operatorname{span}\{\varphi_{ij}: 1 \leqslant j \leqslant m_i\} \backslash \{0\}, \text{ for } i \in I \cap \{j: 1 \leqslant j \leqslant s\}. \tag{2.7}$$

From (2.6) we know that (2.5) is equivalent to

$$\sum_{i\in I} \varphi_i(g(t_i))=0, \qquad g\in G. \tag{2.8}$$

If $I\subset\{j: 1\leqslant j\leqslant s\}$, then (2.8) and (2.7) contradict the fact that $\{\varphi_{ij}\circ\delta_{t_i}: 1\leqslant j\leqslant m_i,\ 1\leqslant i\leqslant s\}$ is linearly independent on G. Hence

$$A\cap\{t_i: i\in I\}=\{t_i: s+1\leqslant i\leqslant r\}\cap\{t_i: i\in I\}\neq\emptyset.$$

This completes the proof of Lemma 3. ∎

Now we can give another characteristic description of the lower semicontinuity of P_G.

LEMMA 4. *The following statements are equivalent*:

(1) P_G *is lsc*;

(2) *For any* $\{t_i: 1\leqslant i\leqslant r\}\subset T$, *if there exist* $\psi_i\in X^*\setminus\{0\}$, $1\leqslant i\leqslant r$, *such that*

$$\sum_{i=1}^{r} \psi_i(g(t_i))=0, \qquad g\in G, \tag{2.9}$$

then for any g *in* G *with* $\{t_i: 1\leqslant i\leqslant r\}\subset Z(g)$, *we have*

$$\{t_i: 1\leqslant i\leqslant r\}\subset \operatorname{int} Z(g). \tag{2.10}$$

Proof. (1)$\Rightarrow$(2). First we claim that there are $x_i\in S(X)$ (the unit sphere of X) and $\varphi_i\in X^*\setminus\{0\}$, $1\leqslant i\leqslant r$, such that

$$\sum_{i=1}^{r} \varphi_i(g(t_i))=0, \qquad g\in G, \tag{2.11}$$

$$\|\varphi_i\|=\varphi_i(x_i), \qquad 1\leqslant i\leqslant r. \tag{2.12}$$

In fact, if dim X is finite, then X is reflexive. Let $x_i\in S(X)$ such that

$$\psi_i(x_i)=\|\psi_i\|, \qquad 1\leqslant i\leqslant r. \tag{2.13}$$

Take ψ_i as φ_i. Then (2.9) and (2.13) imply (2.11) and (2.12).

Now we assume dim $X=\infty$. Since $G_i:=\{g(t_i): g\in G\}$ is a finite-dimensional subspace of X, there is $x_i\in S(X)$ such that

$$1=\|x_i\|_X=d(x_i, G_i), \qquad 1\leqslant i\leqslant r.$$

By the characterization about best approximations [8], there exist $\varphi_i \in X^* \setminus \{0\}$ such that

$$\varphi_i(x_i) = \|\varphi_i\|, \qquad 1 \leqslant i \leqslant r,$$

$$\varphi_i(g(t_i)) = 0, \qquad g \in G,\ 1 \leqslant i \leqslant r,$$

which imply (2.11) and (2.12). Thus our claim is true.

It is not difficult to construct h in $C_0(T, X)$ such that

$$h(t_i) = x_i, \qquad 1 \leqslant i \leqslant r,$$

$$\|h\| = 1.$$

Now for any $g \in G$ with $\{t_i\colon 1 \leqslant i \leqslant r\} \subset Z(g)$, we need to show

$$\{t_i\colon 1 \leqslant i \leqslant r\} \subset \operatorname{int} Z(g). \tag{2.14}$$

Without loss of generality, we may assume

$$\|g\| = \tfrac{1}{2}.$$

Set

$$\rho(t) = \max\{1 + \|g(t)\|_X,\ \|g(t)\|_X + \|h(t) - g(t)\|_X\}, \qquad t \in T.$$

Then it is easy to verify that

$$\rho(t_i) = 1, \qquad 1 \leqslant i \leqslant r,$$

$$1 \leqslant \rho(t) \leqslant 2, \qquad t \in T.$$

Define

$$f(t) = h(t)/\rho(t), \qquad t \in T.$$

Then for any $t \in T$, we have

$$\begin{aligned}\|f(t) - g(t)\|_X &= \|h(t) - g(t) + (\rho(t) - 1) \cdot g(t)\|_X/\rho(t)\\ &\leqslant (\|h(t) - g(t)\|_X + (\rho(t) - 1) \cdot \|g(t)\|_X)/\rho(t)\\ &\leqslant (\|h(t) - g(t)\|_X + \|g(t)\|_X)/\rho(t) \leqslant 1;\end{aligned} \tag{2.15}$$

$$\|f(t)\|_X \leqslant (\|h(t) - g(t)\|_X + \|g(t)\|_X)/\rho(t) \leqslant 1. \tag{2.16}$$

But

$$f(t_i) = x_i, \qquad 1 \leqslant i \leqslant r, \tag{2.17}$$

and we get

$$\|f\|=1. \tag{2.18}$$

By Lemma 2 and (2.11), (2.12), (2.15)–(2.18), we obtain that 0, $g \in P_G(f)$ and

$$\{t_i: 1 \leqslant i \leqslant r\} \subset E(f-P_G(f)).$$

Thus it follows from Lemma 1 that

$$\begin{aligned}\{t_i: 1 \leqslant i \leqslant r\} &\subset E(f-P_G(f)) \\ &\subset \operatorname{int}\{t \in T: p(t)=g(t) \text{ for all } p, g \in P_G(f)\} \\ &\subset \operatorname{int} Z(g-0)=\operatorname{int} Z(g).\end{aligned}$$

This proves (1) ⇒ (2).

(2) ⇒ (1). For $f \in C_0(T, X)$, there is $g^* \in P_G(f)$ [3] such that

$$E(f-P_G(f))=\{t \in T: \|f(t)-g^*(t)\|_X=d(f, G)\} =: E(f-g^*).$$

Set

$$\begin{aligned} V &= \operatorname{int}\{t \in T: p(t)=g(t) \text{ for all } p, g \in P_G(f)\}, \\ A &= E(f-P_G(f)) \setminus V, \\ G_1 &= \{g \in G: V \subset Z(g)\}. \end{aligned}$$

If $A \neq \varnothing$, we claim that

$$\operatorname{card}(A) > \dim G_1|_A / \dim X. \tag{2.19}$$

In fact, if (2.19) fails to be true, then

$$\dim G_1|_A = \operatorname{card}(A) \cdot \dim X. \tag{2.20}$$

Let $T_0 = T \setminus V$, $G_0 = G_1|_{T_0}$, $f_0 = (f-g^*)|_{T_0}$. Then it is easy to verify that

$$P_{G_0}(f_0) = (P_G(f) - g^*)|_{T_0}.$$

So $A = E(f_0)$ and $0 \in P_{G_0}(f_0)$. By Theorem 1 in [8], we obtain that there exist $t_i \in E(f_0) = A$ and $\varphi_i \in X^* \setminus \{0\}$, $1 \leqslant i \leqslant r$, such that

$$\sum_{i=1}^{r} \varphi_i(p(t_i)) = 0, \qquad p \in G_0, \tag{2.21}$$

$$\sum_{i=1}^{r} \varphi_i(f_0(t_i)) = \|f_0\| \cdot \sum_{i=1}^{r} \|\varphi_i\|. \tag{2.22}$$

But from (2.20) we can derive that card(A) and dim X are both finite, and that there is $p^* \in G_1$ satisfying

$$p^*(t) = f(t) - g^*(t) = f_0(t), \qquad t \in A. \tag{2.23}$$

By $t_i \in A$ for $1 \leqslant i \leqslant r$ and $p^* |_{T_0} \in G_0$, we can see that (2.23) contradicts (2.21) and (2.22). Thus (2.19) is true.

Now by (2.19) and Lemma 3, we deduce that there exist $t_i \in A \cup V$ and $\varphi_i \in X^* \setminus \{0\}$, $1 \leqslant i \leqslant r$, such that

$$\sum_{i=1}^{r} \varphi_i(g(t_i)) = 0, \qquad g \in G, \tag{2.24}$$

$$A \cap \{t_i : 1 \leqslant i \leqslant r\} \neq \varnothing. \tag{2.25}$$

But for any p, $g \in P_G(f)$, $E(f - P_G(f)) \subset Z(p-g)$ [3]. So

$$t_i \in A \cup V \subset E(f - P_G(f)) \cup \operatorname{int} Z(p-g) \subset Z(p-g), \qquad 1 \leqslant i \leqslant r,\ p,\ g \in P_G(f). \tag{2.26}$$

By hypothesis (2), equations (2.24), and (2.26), we obtain

$$t_i \in \operatorname{int} Z(p-g), \qquad 1 \leqslant i \leqslant r,\ p,\ g \in P_G(f). \tag{2.27}$$

Since dim G is finite, (2.27) is equivalent to (see [10])

$$t_i \in \operatorname{int}\{t \in T : p(t) = g(t) \text{ for all } p,\ g \in P_G(f)\} = V, \qquad 1 \leqslant i \leqslant r. \tag{2.28}$$

This contradicts (2.25). The contradiction shows that for every $f \in C_0(T, X)$, we have

$$E(f - P_G(f)) \subset \operatorname{int}\{t \in T : p(t) = g(t) \text{ for all } p,\ g \in P_G(f)\}.$$

By Lemma 1, P_G is lsc. ∎

Proof of Theorem 1. *Necessity.* Assume that (1.1) fails to be true. Then there is $p \in G$ such that

$$\operatorname{card}(\operatorname{bd} Z(p)) > \dim\{g|_{\operatorname{bd} Z(p)} : g \in G \text{ and } \operatorname{int} Z(p) \subset Z(g)\}. \tag{2.29}$$

By Lemma 3, we obtain that there exist $t_i \in Z(p)$ and $\varphi_i \in X^* \setminus \{0\}$, $1 \leqslant i \leqslant r$, such that

$$\sum_{i=1}^{r} \varphi_i(g(t_i)) = 0, \qquad g \in G, \tag{2.30}$$

$$\operatorname{bd} Z(p) \cap \{t_i : 1 \leqslant i \leqslant r\} \neq \varnothing. \tag{2.31}$$

Since P_G is lsc, Lemma 4 tells us that (2.30) implies

$$t_i \in \operatorname{int} Z(p), \qquad 1 \leqslant i \leqslant r,$$

which contradicts (2.31).

Sufficiency. By Lemma 4, we know that it is sufficient to show that statement (2) in Lemma 4 is true. Suppose that $t_i \in T$ and $\varphi_i \in X^* \setminus \{0\}$, $1 \leqslant i \leqslant r$, such that

$$\sum_{i=1}^{r} \varphi_i(g(t_i)) = 0, \qquad g \in G. \tag{2.32}$$

Then for any $g \in G$ with $t_i \in Z(g)$, $1 \leqslant i \leqslant r$, we must prove

$$t_i \in \operatorname{int} Z(g), \qquad 1 \leqslant i \leqslant r. \tag{2.33}$$

In fact, if (2.33) is false, we may assume that for some s, $1 \leqslant s \leqslant r$, $\{t_i: 1 \leqslant i \leqslant r\} \setminus \operatorname{int} Z(g) = t_i: 1 \leqslant i \leqslant s\}$. Set

$$G_0 = \{p \in G: \operatorname{int} Z(g) \subset Z(p)\}.$$

Then by (2.32), we have

$$\sum_{i=1}^{s} \varphi_i(p(t_i)) = 0, \qquad p \in G_0 |_{\operatorname{bd} Z(g)},$$

which implies

$$\dim G_0 |_{\operatorname{bd} Z(g)} \leqslant \operatorname{card}(\operatorname{bd} Z(g)) \cdot \dim X^* - 1. \tag{2.34}$$

Since $\operatorname{bd} Z(g) \neq \varnothing$, (1.1) implies that $\dim X = \dim X^*$ is finite. Thus (2.34) contradicts (1.1). This completes the proof of Theorem 1. ∎

Since for any $g \in G$,

$$\begin{aligned} &\dim\{p |_{\operatorname{bd}} Z(g): p \in G \text{ and } \operatorname{int} Z(g) \subset Z(p)\} \\ &\qquad \leqslant \dim\{p: p \in G \text{ and } \operatorname{int} Z(g) \subset Z(p)\} - 1, \end{aligned}$$

the following result is an immediate corollary of Theorem 1:

COROLLARY 2. *If P_G is lsc, then for any nonzero $g \in G$, we have*

$$\operatorname{card}(\operatorname{bd} Z(g)) \leqslant (\dim\{p \in G: \operatorname{int} Z(g) \subset Z(p)\} - 1\}/\dim X. \tag{2.35}$$

Proof of Theorem 2. The necessity is a special case of Corollary 2. Now we show the sufficiency. By Theorem 1, it suffices to show that (1.1) holds for every g in G.

Assume that (1.1) is false, i.e., there is $p \in G$ such that

$$\operatorname{card}(\operatorname{bd} Z(p)) > \dim\{g|_{\operatorname{bd} Z(p)}: g \in G \text{ and } \operatorname{int} Z(p) \subset Z(g)\}. \tag{2.36}$$

Equation (2.36) implies that there is $t^* \in \operatorname{bd} Z(p)$ such that

$$\dim G|_{Z(p)} = \dim G|_{Z(p)\setminus\{t^*\}}. \tag{2.37}$$

For simplicity, we denote

$$G(A) := \{g \in G: A \subset Z(g)\}.$$

Since $\dim G$ is finite, there is an open set V containing t^* such that for any $g \in G$ with $t^* \in \operatorname{int} Z(g)$, we have $V \subset Z(g)$ [10]. Choose $t_i \in T \setminus Z(p)$, $0 \leqslant i \leqslant r$, such that

$$\dim G(Z(p)) = \dim G(Z(p))|_{\{t_i: 0 \leqslant i \leqslant r\}} = r + 1, \qquad t_0 \in V. \tag{2.38}$$

Obviously, there is $q \in G(Z(p))$ satisfying

$$q(t_i) = \begin{cases} 0, & 1 \leqslant i \leqslant r, \\ 1, & i = 0. \end{cases}$$

$t^* \in \operatorname{int} Z(q)$ implies $t_0 \in V \subset \operatorname{int} Z(q)$. This is impossible. So $t^* \in \operatorname{bd} Z(q)$. It follows from (2.37) and $Z(p) \subset Z(q)$ that

$$\dim G|_{Z(q)} = \dim G|_{Z(q)\setminus\{t^*\}}.$$

By (2.38), we get

$$\dim G(Z(q)) = 1.$$

Hence

$$\begin{aligned} \operatorname{card}(\operatorname{bd} Z(q)) &\geqslant 1 + \operatorname{card}(\operatorname{bd} Z(q) \setminus \{t^*\}) \\ &\geqslant 1 + \dim G(\operatorname{int} Z(q))|_{\operatorname{bd} Z(q)\setminus\{t^*\}} \\ &= 1 + \dim G(\operatorname{int} Z(q))|_{\operatorname{bd} Z(q)} \\ &= 1 + \dim G(\operatorname{int} Z(q)) - \dim G(Z(q)) \\ &= \dim G(\operatorname{int} Z(q)), \end{aligned}$$

which contradicts (1.2). ▮

Corollary 1 follows immediately from Theorem 2 and the Michael selection theorem [14].

REMARKS

Remark 1. A condition similar to (1.1) was used by Zukhovitskii and Stechkin [19] to characterize finite-dimensional Chebyshev subspaces of $C_0(T, X)$.

THEOREM D [19]. *Suppose that X is a k-dimensional strictly convex Banach space and G is an N-dimensional subspace of $C_0(T, X)$. Then G is a Chebyshev subspace of $C_0(T, X)$ if and only if every nonzero $g \in G$ has at most n zeros, and for any $t_i \in T$, $x_i \in X$, $1 \leqslant i \leqslant n$, there is $p \in G$ such that $p(t_i) = x_i$, $1 \leqslant i \leqslant n$, where n is the integer which satisfies $n \cdot k < N \leqslant (n+1) \cdot k$.*

We can reformulate Theorem D along the same lines as Theorem 1:

THEOREM 3. *Suppose that X is strictly convex and G is a finite-dimensional subspace of $C_0(T, X)$. Then G is a Chebyshev subspace of $C_0(T, X)$ if and only if for each nonzero g in G, we have*

$$\operatorname{card}(Z(g)) \leqslant (\dim X)^{-1} \cdot \dim\{p\,|_{Z(g)}: p \in G \text{ and } \operatorname{int} Z(g) \subset Z(p)\}. \quad (3.1)$$

When $\dim X$ is finite, Theorem 3 is equivalent to Theorem D; if $\dim X$ is infinite, it can be proved similarly as Theorem 1. Theorem 3 can be considered as the prototype of Theorem 1.

Remark 2. By Lemma 4, we can deduce the following result:

COROLLARY 3. *If P_G is lsc, then for any $T^* \subset T$ and $G^* = G\,|_{T^*}$, P_{G^*} is also lsc.*

Remark 3. Generally, (2.35) is not sufficient for the lower semicontinuity of P_G. Here is a simple counterexample.

EXAMPLE 1. Let X be the two-dimensional Euclidean space R^2, $e_1 = (1, 0)$, $e_2 = (0, 1)$, and $T = [0, 1]$. Define $G := \operatorname{span}\{g_1, g_2, g_3\}$ as follows:

$$g_1(t) = t \cdot e_1, \qquad g_2(t) = t^2 \cdot e_2, \qquad g_3(t) = e_1 + e_2.$$

Then it is easy to check that for any nonzero g in G,

$$\operatorname{card}(Z(g)) \leqslant 1 = (\dim G - 1)/\dim X. \quad (3.2)$$

But for any nonzero g in G, $\operatorname{int} Z(g)=\varnothing$. So, (1.1) and (3.1), (2.35) and (3.2) are equivalent, respectively. However,

$$\operatorname{card}(Z(g_1))=1>\tfrac{1}{2}=\dim G\,|_{Z(g_1)}/\dim X.$$

By Theorem 1, we know that P_G is not lsc.

This example also shows that $\dim G\,|_{Z(g)}$ in (3.1) cannot be replaced by $\dim G-1$.

Remark 4. Unlike the case of lower semicontinuity, the local connectedness of the underlying topological space T plays an important role in the case of continuous selections. We cannot expect a simple form of characterization conditions of G which ensure the existence of continuous selections for P_G. Here we give a simple example which shows that if T is not locally connected, then Theorem C fails to be true.

EXAMPLE 2. Let $I_k=[1-2/4^k, 1-1/4^k]$, $J_k=[-1+3/4^k, -1+4/4^k]$, and $T=\{-1,1\}\cup(\bigcup_{k=1}^{\infty} I_k)\cup(\bigcup_{k=1}^{\infty} J_k)$. Define

$$g_1(t)=1, \qquad g_2(t)=|t|, \qquad t\in T;$$
$$G=\operatorname{span}\{g_1, g_2\}.$$

Then it is easy to verify that every nonzero g in G satisfies conditions (1) and (2) in Theorem C. Let $f(t)=t$ for $t\in T$. Then we can easily prove that

$$P_G(f)=\{\lambda(g_1-g_2): |\lambda|\leqslant 1\}, \tag{3.3}$$

$$E(f-P_G(f))=\{-1,1\}. \tag{3.4}$$

It follows from (3.3) and (3.4) that there is no g in $P_G(f)$ satisfying the following condition:

$$E(f-P_G(f))\subset \operatorname{int}\{t\in T: (f(t)-g(t))\cdot(g(t)-p(t))\geqslant 0\}, \qquad \text{for } p\in P_G(f).$$

By Theorem B, we know that P_G has no continuous selection.

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